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SEPARATION-FREE FLOW AROUND A BLUNT BODY BY A HIGH-VELOCITY ELASTIC-PLASTIC FLOW*

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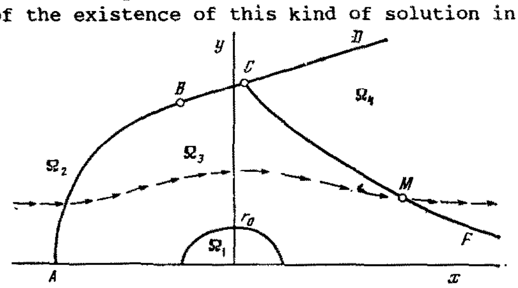
A flow problem in which the well-known Lavrent'ev assumption about the closeness between the velocity and stress fields in the high-velocity motion of a rigid body in a solid medium and the analogous fields from the solution of the hydrodynamic problem /1/ is translated into the language of asymptotic representations, is considered. The ratio between the yield point τ_s and the velocity head is the small parameter. The unknown boundaries of the elastic and plastic zones and the unloading wave front are found in a zero approximation (the hydrodynamic problem). The dimensions of the plastic zone during flow around the body turns out to equal $(\tau_s/\mu)^\alpha$, where $\alpha = 1/2$ for a cylinder and $\alpha = 1/3$ for a sphere and μ is the shear modulus.

A solution of the boundary-layer type that overcomes the shortcoming in complying with the plastic flow condition, is examined near the body. A first approximation is constructed in the whole flow domain. Formulas are presented for the strength corrections to the maximum normal stress and the force acting by the flow on a sphere or cylinder.

1. *Formulation of the problem.* An incompressible elastic-plastic infinite stream flows around a rigid body with a smooth surface S in the steady-state mode. The body is at rest in a Cartesian Euler xyz coordinate system, and the x-axis coincides with the flow direction at infinity. The flow far ahead of the body $(x \to -\infty)$ is homogeneous, is not sheared, and the pressure p_∞ therein is considered to be sufficient so that the flow is separation-free, with velocity c, where $c^2 \gg b^2$ with b the velocity of the transverse elastic waves.

We realize the following a priori flow scheme that is symmetric relative to the x-axis (see the figure) for the plane and axisymmetric problems. Far from the body Ω_1 the material is in the elastic and irrotational state (the zone Ω_2). Passage to the plastic (vortex) state is realized at the front of a transverse shock ABCD. The particles of the medium behind the front section ABC continue to remain in this state until the reverse passage into the elastic state on the unloading wave CF in the domain Ω_3. The particles are unloaded elastically at once in the zone Ω_4. Tangency of the curvilinear zone boundary AB of the zone and the normal or conical transverse wave front making an angle $\alpha = b/c$ with the x-axis occurs at the point B. The boundary of the unloading zone starts from a certain point C on the front BD. Let us emphasize a peculiarity of the wave CD. Unlike the usual elastic transverse wave, the jump in the shear stresses thereon is constrained by the plasticity condition. Particles intersecting the front CD arrive in the plasticity state only at the front. The possibility of the existence of this kind of solution in the neighbourhood of the BC and CD wave fronts can be shown by simple model problems.



The curved boundaries are not known. Moreover, the velocity field $\mathbf{u}$, the pressure p, the stress deviators $\mathbf{T} = \{\tau_{ij}\}$, that are normalized by the velocity c, the velocity head ρc^2 and the yield point τ_s in the Saint-Venant-Mises plastic model are sought. We select half the characteristic dimension of the body section r_0 (use the figure) as the unit of length.

The mathematical problem of determining all the unknown elements consists of solving the equations of incompressibility and conservation of momentum outside the domain Ω_1, Hook's law in the elastic zones Ω_2 and Ω_4, the Saint-Venant-Mises plasticity equation in

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the domain Ω_3 taking the matching conditions into account for the solutions on the transverse shock *ABD* and the unloading wave *CF*, the laws of non-penetration and friction on the surface *S*, the behaviour at infinity, and the symmetry conditions on the x-axis:

$$\operatorname{div} \mathbf{u} = 0, \quad \sigma_{ii} = -p + f^2\tau_{ii} \tag{1.1}$$

$$\frac{d}{dt}\mathbf{u} = \frac{1}{2}\operatorname{grad}(\mathbf{u})^2 + \operatorname{rot}\mathbf{u}\times\mathbf{u} = -\operatorname{grad} p + f^2 \operatorname{div}\mathbf{T} \quad (\Omega_2 \cup \Omega_3 \cup \Omega_4) \tag{1.2}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^*), \quad \mathbf{E} = \int\limits_l \boldsymbol{\varepsilon}\frac{dl}{|\mathbf{u}|}$$

$$\mathbf{T} = \frac{1}{m}\int\limits_{-\infty}^{x} \boldsymbol{\varepsilon}|_{y=\mathrm{const}}\, dx, \quad J(\mathbf{T}) = \frac{1}{2}\tau_{ij}\tau_{ij} < 1 \quad (\Omega_2) \tag{1.3}$$

$$\mathbf{T} = \mathbf{T}(M_*) + \frac{1}{m}\int\limits_{x_*}^{x} \boldsymbol{\varepsilon}|_{y=\mathrm{const}}\, dx \quad (\Omega_4) \tag{1.4}$$

$$\mathbf{T} = \boldsymbol{\varepsilon} J^{-1/2}(\varepsilon) \Rightarrow J(\mathbf{T}) = 1 \quad (\Omega_3) \tag{1.5}$$

$$[u_n] = [\sigma_{nn}] = 0, \quad [u_\tau]\, u_n = f^2 [\tau_{n\tau}] \quad (ABD) \tag{1.6}$$

$$[J[\mathbf{T}] = 0] \qquad (AB) \tag{1.7}$$

$$\partial/\partial x J(\mathbf{E}) = 0, \quad (C), \quad \mathbf{T}|_{CD} = \mathbf{T}|_C \quad (CD) \tag{1.8}$$

$$\partial/\partial x J[\mathbf{E}] = [\mathbf{u}] = [p] = 0 \quad (CF) \tag{1.9}$$

$$u_n = 0, \quad \tau_{n\tau} = -1 \quad (S) \tag{1.10}$$

$$\mathbf{u} = (1, 0, 0), \quad p = p_0, \quad \mathbf{T} = 0 \quad (x \to -\infty) \tag{1.11}$$

$$u_n = 0, \quad \tau_{n\tau} = 0 \quad (y = 0,\ x, y \notin \Omega_1)$$

$$f^2 = \tau_s/(\rho c^2), \quad m = \tau_s/(2\mu)$$

Here n, τ are the normal and tangent to the curve mentioned, M_* is a certain point on the unloading fronts *CF* and *CD*, x_* is its coordinate on the x-axis, the asterisk denotes the matrix transposition operation, and ρ is the density of the medium.

The integrals in (1.2) and (1.3) are taken between $-\infty$ along the streamline l and the running point and are understood to be integrals of generalized functions.

Hooke's law (1.3),(1.4) is written in the form of an approximate relation between the deviator **T** and the strain rate tensor $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\}$ for steady-state motion. Integration in the exact expression for this law should be performed along the streamline.

The problem under consideration will be studied for the following constraints on the parameters

$$f^2 \ll 1, \quad m \ll 1, \quad f^2 m^{-1} \ll 1 \tag{1.12}$$

Physically the constraints (1.12) mean that the strength is secondary in the whole flow domain, the pressure is the principal part of the stress, and the flow is almost hydrodynamic. The condition of being separation-free, that is questionable at first glance, is actually natural. It is realized for an ideal fluid even when the equality $p_\infty = 0$ is satisfied. It can be expected that for $f \ll 1$ small pressure values at infinity will be sufficient for the cavern behind a smooth convex body to collapse. But even when a small zone of flow separation from the body appears, the drag force obtained because of the strength of the medium will not change substantially. It can be used to correct the hydrodynamic drag force that will already be non-zero.

The condition $m \ll 1$ is satisfied for the majority of materials, while the last inequality is equivalent to the high velocity condition $b^2/c^2 \ll 1$ – the flow is supersonic in the transverse waves. The streamlines in the elastic zone turn out to be remote from the body and practically straight lines, and consequently, the approximate taken in (1.3), (1.4) will be justified.

The incompressibility can be taken since shears, and not changes in the particle volumes, are more important in a flow. Moreover, this enables us to solve problems with final formulas. In practical situations a somewhat larger value of the density should be taken than in the unloaded state of the medium, the density of a packed material.

Relationships (1.5) reflect the ideally plastic behaviour of a material without hardening and without taking account of the contribution of the elastic strains to the tangential stresses. Although this is the simplest model it correctly reflects the substance of plasticity: in practice the unbounded growth of strain for a limited level of deviator stress components.

It follows from the condition of continuity of the second invariant (1.7) and the plasticity condition (1.5) that the plasticity condition is satisfied on the line *AB* on the elastic zone side.

It is assumed that particles of the medium slide along the surface S, where there are no explicit constraints on the tangential velocity component (as for a viscous fluid, say). The plasticity condition is then formulated on the boundary S. It is assumed that the pressure is sufficiently large because of the high-velocity nature of the flow and the presence of the pressure p_∞ at infinity so that the dry friction law changes to the plastic flow law on the body surface /2/.

The conditions for the beginning of unloading at once behind the shock front (1.8) (they are to determine the location of the point C) and on the line CF (1.9) require a knowledge of the finite strain tensor $\mathbf{E} = \{e_{ij}\}$. The rule for calculating this tensor is indicated in (1.2) and the jump in the tensor $\mathbf{E}$ on passing through the shock should be taken into account.

Therefore, the problem of the stationary flow around a rigid body by a high-velocity elastic-plastic medium is to solve the complex non-linear problem (1.1)-(1.11). The non-linearity appears in the presence of the unknown boundaries AB and CF, in (1.2), (1.5) in conditions (1.6), (1.5) and (1.9). It is impossible to obtain lucid results for the application of asymptotic approaches without awkward computations or simplifying assumptions.

The development of an idea of Lavrent'yev is the key to solving the problem in this latter aspect. It consists of the fact that the strength properties of a medium only yield corrections to the hydrodynamic results for the high-velocity strain of solids, where these corrections are only essential in certain cases, e.g., when determining the flow forces acting on the body, say. The determination of such corrections is in fact the content of this paper. An asymptotic expansion of the solution is performed in the small parameter f. Two terms of the expansion are found, that corresponding to $f = 0$ and the next in order of magnitude.

2. *The zeroth approximation.* As is seen from (1.1)-(1.11), for $f = 0$ the number of equations and the number of unknowns are not reduced. Only the order of the equation of conservation of momentum is reduced and the condition of continuity of the hydrodynamic velocity field $\mathbf{V}$ and the pressure q outside the body in the flow follows. Only the non-penetration condition is satisfied on the surface S. Satisfaction of the friction condition from (1.10) is a problem of the next approximation. The question of determining the functions $\mathbf{V}$ and q is a problem of the separation-free flow around a solid body a steady non-rotational ideal incompressible fluid flow. We shall later use expressions known from hydrodynamics for the quantities $\mathbf{V}$ and q and concentrate our efforts on obtaining strength corrections to the hydrodynamic fields.

A knowledge of the velocity $\mathbf{V}$ in the zeroth approximation will enable as to determine the stress deviator $\mathbf{T}$ by means of (1.2)-(1.5) and the zone boundary. The corollary $J(\mathbf{T}_e) = 1$ of conditions (1.5) and (1.7) is used to seek the front AB ($\mathbf{T}_e$ is the stress deviator in the elastic domain). At the point B from which a plane conical front extends, the slope of the curve AB is made equal to α. The first condition in (1.9) is for determining the unloading wave front CF. The deviator $\mathbf{T}$ has a discontinuity on the boundary ABD. For $f \neq 0$ a discontinuity in the normal stresses is not allowable because of incompressibility. Cancellation of this discontinuity is realized when constructing the first approximation.

3. *The plane problem in the zeroth approximation. The flow around a cylinder.* Following /3, 4/, we introduce the complex velocity $v(z) = v_x - iv_y$, which is an analytic function of the complex variable $z = x + iy = re^{i\theta}$. For the problem of the flow round a circle of unit radius and an ellipse with semi-axes a, 1

$$v(z) = 1 - z^{-2}, \quad v(z) = \frac{1}{a-1}\left(1 - \frac{z}{\sqrt{z^2 - a^2 + 1}}\right) \tag{3.1}$$

We introduce the complex rate of strain $\varepsilon(z)$ and the complex function $\tau(z)$, not generally analytic, by means of the formulas

$$\varepsilon(z) = \varepsilon_{xx} - i\varepsilon_{xy} = v'(z), \quad \tau(z) = \tau_{xx} - i\tau_{xy}, \quad J(T) = |\tau(z)|^2 \tag{3.2}$$

We first present general expressions for the functions $\tau(z)$ and the pressure q by starting from the relationships (1.3), (1.5), (3.2) and the Bernoulli integral

$$\tau(z) = m^{-1}(v(z) - 1) \quad (\Omega_2) \tag{3.3}$$

$$\tau(z) = \sqrt{v'(z)/v'(\bar{z})}, \quad J(\varepsilon) = \varepsilon(z)\,\varepsilon(\bar{z}) \quad (\Omega_3)$$

$$q = p_0 + {}^1/_2(1 - |v(z)|^2) \quad (\text{outside of } \Omega_1)$$

and then the formulas for the stresses in the case of a cylinder in the elastic and plastic zones

$$\tau_{rr} = -1/(mr^2), \quad \tau_{r\theta} = 0 \quad (\Omega_2) \tag{3.4}$$
$$\tau_{rr} = \cos\theta, \quad \tau_{r\theta} = \sin\theta \quad (\Omega_3)$$

We note that the stresses $\tau_{rr}, \tau_{r\theta}$ are independent of r in the plastic zone and the D'Alembert paradox holds for the zeroth approximation: the drag force equals zero. We have the equation $|v(z) - 1| = m$ to determine the elasticity and plasticity zone boundaries from (1.5), (1.7) and (3.2).

For the flow around a circle and an ellipse (if we neglect quantities of the order of $\sqrt{m\,|a^2 - 1|}$ compared with unity), this will be a circle with centre coincident with the centre of the streamlined figure and the radii

$$R = m^{-1/2} = \sqrt{2\mu/\tau_s}, \quad R = \sqrt{(a+1)/(2m)}$$

Because of the conditions (1.12) we have $R \gg 1$. The location of the front BD (see the figure) is found at once. The point B has the coordinates $z_B = R\exp[i(\alpha + \pi/2)]$. Thus, the characteristic dimensions (and shape) of the plastic zone are found and it is shown that they are large compared with the body dimensions.

In the neighbourhood of the stagnation point $z = -1$ we have an approximate expression for the velocity: $v(z) \approx -r(z+1)$. This means that a particle on the streamline $y = 0, x < 1$ $(\psi = 0)$ at a finite distance from the stagnation point at a certain time will reach the latter in an infinite time. The infinite time is required by the particle and by the fact that it departs from the stagnation point. The very same occurs around the point $z = +1$. It is clear that a particle moving along a very nearly streamline $0 < \psi \ll 1$, will pass around the cylinder in a finite time and depart further. We now take a particle far upstream in the form of a tiny square with sides parallel to the axes and middle line on the x axis. After a long time its middle line comes extremely close to the stagnation point while the upper and lower sides, moving along their streamlines, traverse a circle, the particle is transformed into a strange, strongly elongated figure. Particles having no common points with the x axis will be deformed less, but the closer they are to the x-axis, the more strongly they will be elongated. This illustrates the fact that deformations in the flow are large. We will compute them for a cylinder

$$E(z) = e_{xx} - ie_{xy} = \int_l (\varepsilon(z)\,|v(z)|^{-1})\,|_{\psi=\text{const}}\, dl = \int_{-\infty}^{\varphi} (\varepsilon\,|v|^{-2})_{\psi=\text{const}}\, d\varphi \tag{3.5}$$

Here φ, ψ are the velocity potential and stream function.

For $x = \pm\infty$, we will have $\psi = y$ and the "residual" strains of the particles in the stream can be found by evaluating the integral. It is expressed in terms of the complete elliptic integral

$$E^\infty(y) = -2i\left\{\operatorname{sign} y \cdot \frac{\pi}{2} + y\left[\frac{2}{k}E(k) + \left(\frac{k}{2} - \frac{2}{k}\right)K(k)\right]\right\} \tag{3.6}$$
$$k^2 = 4(4 + y^2)^{-1}$$
$$E^\infty = -\pi i \quad (0 < y \ll 1), \quad E^\infty = -\frac{3\pi}{4}\frac{i}{y^4} \quad (y \gg 1)$$

It follows for limit values of the strain from the upper half-plane and for $|z| \gg 1$ from (3.3), (3.5), and (3.6) that

$$e_{xx}^\infty(+0) = 0, \quad e_{xy}^\infty(+0) = \pi \tag{3.7}$$
$$E(z) = E^\infty(y) + v(z) - 1, \quad |z| \gg 1$$

The equation for the unloading zone boundary is found from the first condition of (1.9) by assuming that it holds for $|z| \gg 1$

$$\operatorname{Re}[\overline{\varepsilon(z)} \cdot E(z)] = 0 \tag{3.8}$$

Using the results (3.1), (3.2), (3.5) and (3.7) it reduces for large and small values of y to the form

$$\sin 2\theta = {}^3/_4\pi\,(1 + 2\cos 2\theta)\,y^{-2} \Rightarrow \theta \approx {}^1/_2\pi, \quad y \gg 1$$
$$\pi r^2 \sin 3\theta = 2\cos\theta \Rightarrow \theta \to 0, \quad r \to \infty, \quad y \ll 1$$

Now, the curve $y(x)$, the unloading zone boundary, can be described qualitatively. It starts from the point $z_C \approx im^{-1/2}$ for $x = 0$, decreases monotonically as the coordinate x increases (this can be shown), and $y = 0$ for $x = \infty$. We conclude from the above that the unloading zone boundary is a large distance away from the body of the order of $m^{-1/2}$. The front CD is also an unloading zone boundary.

We obtain the solution for the complex stress $\tau(z)$ in the unloading zone from (1.4) and (3.2)

$$\tau(z) = \tau(z_*) + \frac{1}{m^{-1}}(v(z) - v(z_*)), \quad z_* = r_* \exp(i\theta_*)$$

Here z_* is a point on the unloading zone boundary. According to the results (3.4) and (3.8) and the condition on the deviator (1.8) the relations

$$\tau_*(z_*) = \exp 3i\theta_* \quad (y \leqslant m^{-1/2}), \quad \tau(z_*) = -i \quad (y > m^{-1/2})$$

will hold for $y \leqslant m^{-1/2}$ and $y \geqslant m^{-1/2}$.

The construction of the zeroth approximation in the problem for the flow around a circle is completed. The solution for other bodies can be obtained by analogy with the above exposition.

4. The zeroth approximation in the problem of the flow around a sphere. The zeroth approximation in the axisymmetric, or generally spatial, case is constructed in the same way as for plane strain. However, in technical respects it is more complicated to do this. Consequently, we will confine ourselves to relying on the results of investigating fields in the elastic and plastic zones.

The following are the expressions for the velocity potential, the particle velocity, the pressure, the stress $J(\varepsilon)$ and the radius of the plastic zone

$$\Phi = [r + (2r^2)^{-1}]\cos\theta, \quad \mathbf{v} = \operatorname{grad}\Phi, \quad q = p_0 + {}^1/_2(1 - \mathbf{v}^2) \tag{4.1}$$
$$-\tau_{rr} = 2\tau_{\theta\theta} = 2\tau_{\beta\beta} = (mr^3)^{-1}, \quad \tau_{r\theta} = 0 \quad (\Omega_2)$$
$$-{}^1/_2\tau_{rr} = \tau_{\theta\theta} = \tau_{\beta\beta} = -d\cos\theta, \quad \tau_{r\theta} = d\sin\theta, \; d = (1 + 2\cos^2\theta)^{-1/2}$$
$$J(\varepsilon) = {}^9/_4 r^{-8} d^{-2}, \quad R = ({}^3/_4 m^{-2})^{1/8}$$

We have for the streamline $\xi = \text{const}$ /3, 4/

$$(r^2 - r^{-1})\sin^2\theta = \xi = \text{const}, \quad \beta = \text{const}$$

Here r, β, θ is a spherical system of coordinates with origin at the centre of the sphere, and the angle θ is measured from the x-axis.

The radius of the plastic zone is less than in the plane problem, however, even in the case of a sphere $R \gg 1$. The stress deviator $\mathbf{T}$ in the plastic zone is also independent of r. The location of the conical front BD is determined as in Sect.3.

5. *On an inner expansion (boundary layer).* The purpose of constructing a boundary-layer type solution is to find the stress deviator $\mathbf{T}$ from (1.5) for $f \ll 1$ and from the conditions (1.10). The friction condition would not be satisfied in the zeroth approximation, consequently, corrections $O(1)$ to the tensor $\mathbf{T}$ should be expected. At the same time, it is shown in /5, 6/* (*See also Flitman, L.M., On the boundary layer in certain problems of the dynamics of a plastic medium. Preprint No.150, Inst. of Problems of Mechanics, Academy of Sciences of the USSR, Moscow, 1980.) that corrections to the velocity will be small of the order of f. The properties of solutions of the flow problems under consideration thereby differ radically from the properties of solutions in viscous fluid dynamics, where the correction to the velocity is $O(1)$ near the body. This difference is due to the fact that particle slip around the body is achieved during flow by a plastic flow, but there is a constraint on the magnitude of the shear stress in the form of the Mises plasticity condition. Adhesion is required for a viscous fluid in the classical formulation and there is no constraint on the tensor $\mathbf{T}$.

The general equations of non-stationary plastic flow near a rough surface S (of boundary-layer type) are obtained and investigated in /5, 6/. Even in cases of the simplest body geometry their exact solutions are not found successfully and it is necessary to rely on electronic computer calculations. However, the following facts /5, 6/ are sufficient for the purposes of this investigation: in principle, corrections of the above-mentioned order exist for the stresses and velocities of boundary-layer type (i.e., they vary rapidly around the surface S and decrease to zero with distance from the body), the boundary layer thickness is of the order f, the asymptotic approach being used contains no internal contradictions, the normal stress on areas parallel to the surface of the streamlined body varies slightly, and therefore, their values from the external expansion can be utilized to determine these stresses on the body (there are ready formulas for the shear stress on a body in the form of the second condition in (1.10)).

6. Outer asymptotic expansion. First approximation. It follows from the previous results that values of components of the tensor T and the location of the zone boundaries can be considered found on retaining terms not smaller than $O(f^2)$ in expressions for the desired quantities $\mathbf{u}, p, \mathbf{T}$. Moreover, it became known that the stress σ_{nn}, to $o(f^2)$ accuracy, does not change on intersecting the boundary layer.

Let us determine the corrections to the velocity and pressure in the outer asymptotic expansion. We then obtain the uniform asymptotic expansion of all the desired functions to $o(f^2)$ accuracy. The corrections under discussion are of the order of f^2. We set

$$\mathbf{v} = \mathbf{V} + f^2\mathbf{W}, \quad P = q + f^2 s \tag{6.1}$$

The usual process of substituting the representations (6.1) into the fundamental equations and conditions (1.1)-(1.11) and linearizing leads to the following equations and conditions for the corrections in the whole flow domain:

$$\operatorname{div} \mathbf{W} = 0, \quad \nabla P + [\operatorname{rot} \mathbf{W} \times \mathbf{V}] = \operatorname{div} \mathbf{T}; \quad P = s + \mathbf{V}\cdot\mathbf{W} \tag{6.2}$$

$$[W_n] = 0, \quad [\mathbf{W}_\tau] = [\boldsymbol{\tau}_{n\tau}]/v_n, \quad [s] = [\tau_{nn}] \Rightarrow$$
$$[P] = v_n^{-1}\{\mathbf{V}_\tau\cdot[\boldsymbol{\tau}_{n\tau}] + v_n[\tau_{nn}]\} \quad (ABD) \tag{6.3}$$

$$[\mathbf{W}] = 0, \quad [s] = 0 \Rightarrow [P] = 0 \quad (CF); \quad w_n = 0 \quad (S) \tag{6.4}$$

$$\mathbf{W} = s = P = 0 \quad (x \to -\infty) \tag{6.5}$$

Relations (6.2) are a linearized system of equations of ideal fluid dynamics, where $\operatorname{div}\mathbf{T}$ plays the part of the given mass forces.

In the elastic zone $\operatorname{div}\mathbf{T} = 0$ since the flow ahead of the transverse wave front is irrotational while a vortex field occurs only as this front.Then it follows from the second equation in (6.2) and the condition for the function P from (6.5) that $P = 0$ in the elastic zone. The jump of this function on the wave front ABD is given by the fourth condition in (6.3), which means that from the Bernoulli integral along the arc l of the streamline of the field $\mathbf{V}$

$$\partial P/\partial l = |\mathbf{V}|^{-1}(\mathbf{V}\cdot\operatorname{div}\mathbf{T}) \tag{6.6}$$

the function P is defined completely by a quadrature.

We then use the second equation in (6.2) to determine the vortex $\boldsymbol{\Omega} = \operatorname{rot}\mathbf{W}$. Let us project this equation in two different directions that do not coincide with the direction of the streamline at each point of the domain. We obtain two finite linear relationships connecting the components of $\boldsymbol{\Omega}$. Let us append the condition $\operatorname{div}\boldsymbol{\Omega} = 0$ to them. A projection in one direction is sufficient in the plane and axisymmetric case since the vector $\boldsymbol{\Omega}$ has just one non-zero component. The vortex of the desired field $\mathbf{W}$ thereby turns out to be expressed in terms of P and $\mathbf{T}$. Taking account of the first equation in (6.2) and the conditions (6.3)-(6.5), we arrive at the problem of determining a field by means of its vortex and divergence and by the conditions just mentioned.

*7. **The stresses and forces acting on a body in a flow.*** We have the following expressions for P on the axis of symmetry and on the surface of a body from an examination of the Bernoulli integral (6.6) on the streamline $\psi = 0$ ($\theta = 0, \pi$, $r > 1$; $0 < \theta < \pi$, $r = 1$) together with the condition $P = 0$ for $r > R$, $\theta = \pi$, the condition on the jump of the function P on the wave front from (6.3) and from the previous results:

$$P(r, \pi) = 3\ln(R/r), \quad P(1, \theta) = 3(\ln R - 1 - \cos\theta) \quad \text{(a cylinder)} \tag{7.1}$$
$$P(r,\pi) = 8\sqrt{3}\ln\left(\frac{R}{r}\right), \quad P(1,0) = 8\sqrt{3}\ln R - 2\int\limits_{-1}^{\cos\theta}\frac{2+5y^2}{(1+2y^2)^{3/2}}\,dy \quad \text{(a sphere)}$$

A formula can now be obtained for the dimensional normal stress at the frontal stagnation point, maximum in absolute value

$$-\sigma_{rr} = p_0 + \frac{\rho c^2}{2} + \tau_s \times \begin{cases} 3\left(\frac{1}{2}\ln\frac{2\mu}{\tau_s} + 1\right) & \text{(a cylinder)} \\ \frac{2}{\sqrt{3}}\left(8\ln\frac{\sqrt{3}\,\mu}{\tau_s} + 1\right) & \text{(a sphere)} \end{cases} \tag{7.2}$$

The desired strength correction to the hydrodynamic pressure is given by the last component in these expressions. Starting from the results (7.1), the last equality in (6.2), and expression (7.4) for σ_{rr}, analogous expressions can be obtained for σ_{rr} at the rear stagnation point.

To evaluate the force of the flow on the cylinder and sphere by the formula

$$F = 2\int_0^\pi (\sigma_{rr}\cos\theta - f^2\tau_{r\theta}\sin\theta)_{r=1} B(\theta)\,d\theta \tag{7.3}$$

$B(\theta) = 1$ (cylinder), $B(\theta) = \pi\sin\theta$ (sphere)

we have expressions for the stresses for $r = 1$

$$\sigma_{rr} = -q + f^2(\tau_{rr} - s), \quad \tau_{r\theta} = -1 \tag{7.4}$$

$$s = P - \mathbf{W}\cdot\mathbf{V} = 3(\ln R - 1 - \cos\theta) + 2\sin\theta\, w_\theta$$

$$\tau_{rr} = \cos\theta \quad \text{(a cylinder)}$$

$$s = 8\sqrt{3}\ln R - 2\int_{-1}^{\cos\theta} \frac{2+5y^2}{(1+2y^2)^{3/2}}\,dy + \frac{3}{2}\sin\theta w_\theta$$

$$\tau_{rr} = \frac{2\cos\theta}{1+2\cos^2\theta} \quad \text{(a sphere)}$$

The contribution of the quantity q equals zero (D'Alembert's paradox). The still unkown function $w_\theta(1, \theta)$ remains in (7.3) and (7.4). We will use the method of series expansion in trigonometric functions of the angle θ in the problem of determining the fields Ω and **w** and we note that only the second Fourier component $w_\theta^{(2)}$ is necessary to find the force. Omitting the long discussions and computations for analysing the fields P, Ω and **W**, we present the final results obtained to accuracy $O(R^{-1})$

$w_\theta^{(2)} = A\sin 2\theta$; $A = -4.907$ (a sphere) $A = 3$ (a cylinder)

We calculate the equivalent stresses by (7.3) and represent it in the dimensional form

$$F = Br_0^2\tau_s + O(1/R)$$

$B = 4 + 7\pi$ (a cylinder of radius r_0 and length r_0), $B = 55.2$ (a sphere of radius r_0).

Comparing the ratio between the force and the area of the middle section of the body in the flow in the plane and spatial cases, we conclude that the relative difference is 27%. At the same time, the ratio between the strength correction and the maximum normal stress in the planar and spatial cases depends on m and changes from 3.3 for $m^{-1} = 10$ to 4.5 for $m^{-1} = 200$. Results and achieved with less labour in the planar problems than in the spatial problems, and consequently, the comparison of the solution obtained can be used to correct the planar problem solutions.

For applications it is important to indicate the dimensions and shape of the plastic zone since the passage into the plastic state can be identified with fracturing of the medium. The plastic zone can be represented approximately in problems of the flow around a cylinder (sphere) as a semi-infinite body consisting of a semicircle (a hemisphere) of radius $R = m^{-1/2}$ ($R = m^{-1/3}$) and a half-strip (a cylinder) behind it of the same dimensions. A hypothesis is equally likely that the plastic zone will be analogous for other smooth bodies with characteristic dimensions that are not radically different. It has been shown for an ellipse that the zone dimension is proportional to the root of half the sum of its axes, which confirms the hypothesis. The ratio between the plastic zone dimensions in the plane and spatial cases equals $m^{-1/6}$. This means that solutions of planar problems with a correction according to the last formula can be utilized to estimate the dimensions of the plasticity zone for moderate values of m.

Remarks. 1°. If the assumption $c^2/b^2 \gg 1$ is replaced by the condition $c/b < 1$ then the elastic velocity field will contain a vortex part. A transverse shock and the domain of intense vortex motion vanish. The dimensions of the plastic domain increases, but the field in the neighbourhood of the streamlined body, particularly in the boundary layer, varies only slightly. Such assertions enable us to examine the selfsimilar problem of a cylinder expanding and moving along the axis, surrounded by a plastic medium /5/.

Let us emphasize that the boundary-layer representation depends only on the zeroth approximation and, consequently, it is conserved completely as the ratio c/b changes. Therefore, slight sensitivity of the drag force to such changes can be expected. The maximum normal stress on the body surface (7.2) changes because of the growth of the plastic domain, but weakly, since it depends logarithmically on the dimension R of this domain.

2°. On changing to a more complex plastic model, for instance, the Prandtl-Reuss model /7/, the location of all the zone boundaries is conserved for $c/b \gg 1$ because it is determined only by the zeroth approximation. The deviator **T** in the elastic zone retains its value. In the plastic zone it changes and will be determined from the following Prandtl-Reuss equations written for the steady flow of an incompressible medium along the arc l on the zeroth approximation streamline:

$$m\frac{\partial\tau_{ij}}{\partial l} + \lambda\tau_{ij} = \varepsilon_{ij}, \quad \lambda = {}^1/_2\tau_{nm}\varepsilon_{mn}$$

It hence follows that the Saint-Venant-Mises equations utilized in this paper will be satisfied to within a small parameter m. This parameter will be substantial only at a significant distance from the body boundary. The transverse shock vanishes there and stresses on the zone boundaries are made continuous. However, the flow will be close to that considered near the body and the results obtained for the stresses and forces acting on the body can be used for an approximate estimate of the real quantities.

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THE THEORY OF THE FRACTURE OF A SUPERCONDUCTOR IN A MAGNETIC FIELD*

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The stress-strain state of a superconductor in a static magnetic field is investigated from the point of view of the possibility of fracture. Only one force factor is taken into account, namely, the interaction between the field and the surface currents generated by the magnetic field (the Meissner effect /1/). When there are stress concentrators present (corner points, microcracks, inclusions etc.) comparatively weak magnetic fields, for which the specimen does not lose its property of ideal superconductivity, may turn out to be dangerous /1-3/. However, the formulation of the problem remains correct when the superconductor transforms into the normal phase (or simply for a normal conductor) in a variable intense magnetic field under skin-effect conditions and in a quasistatic mechanical state. In this case $t_1 \gg t_2$ is the condition of quasistatics, where t_1 and t_2 are the characteristic times of variation of the magnetic field and the range of wave deformation (volume or shape) of the characteristic dimension of the specimen. Moreover, when there are many factors present, this makes the problem a multiparametric one and extremely complicated to analyse, a preliminary investigation of the effect of each of these factors separately is advisable. The properties of the solutions of plane problems are analysed in detail, in particular, using the examples of regions of

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